Of Analemmas, Mean Time and the Analemmatic Sundial

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There is an interesting irony in the fact that the analemma ('figure 8') curve has become a familiar feature on the classical sundial over the last century and a half, but has only rarely been seen on the analemmatic sundial.[1] One might expect the similarity in names to suggest more of a kinship between the dial and the curve. The purposes of the present article are to consider this irony - a consideration which requires something of an etymological journey - and to elaborate on the design of a standard-time analemmatic sundial which reinforces the kinship by reuniting the dial and the curve.

History and Etymology

In order to proceed, we need to understand the concept of the analemma in a more general setting. Not only is the word *analemma* seldom used today outside of the 'gnomonic community', but when it is used, its meaning tends to be only a narrow derivative of its original sense:

> The word *analemma* means much the same as *lemma*; the analemma is for graphical constructions what the lemma is for geometrical demonstrations; it is a subsidiary figure which is *taken up* to shorten and facilitate the construction of the principal figure. [2]

The particular analemmas which in ancient times proved to be of most use in the design of sundials appear in the works of Vitruvius and Ptolemy. Writing in the first century B.C. in *De Architectura*, the Roman engineer Marcus Vitruvius Pollio noted that "in order to understand the theory of these dials, one must know [the theory] of the analemma".[3] However, the analemma to which he referred was not the now familiar curve relating apparent and mean time. What Vitruvius alluded to was *a graphical procedure equivalent to what is known today as an orthographic projection*. Although he did not provide instructions for its use,[4] Vitruvius made it clear that the analemma was at the core of the ancient practice of sundials.

Early in the second century A.D., Claudius Ptolemaeus wrote *De Analemmata*, a more detailed presentation[5] of a method for projecting the principal circles of the celestial sphere onto a plane - the projection being from a point at an infinite distance along a line perpendicular to that plane. After describing the coordinate system resulting from his projection, Ptolemy presented two distinct methods for determining the coordinates; one method was trigonometric, the other was nomographic - basically, he invented an instrument. This instrument - Ptolemy's analemma - was composed of two pieces: a carpenter's square[6] and a plate of wood or metal with inscribed scales and curves. It allowed one to read values of coordinates directly from the analemma diagram by use of the carpenter's square as a straight-edge.

The analemma is thus also *an instrument which implements a graphical procedure*. This sense of the word is apparent in such references as Regiomontanus' 15th century introduction of a "universal rectilinear analemma"[7] - now generally implemented as an altitude dial on a card; St. Rigaud's publication of his version of that dial as a *New Analemma*,[8] and John Twysden's 1685 *Use of the Great Planisphere called the Analemma*. Note also Valentin Pini's 1598 discussion of Ptolemy's work, in which he introduced his own analemma - a simple armillary dial.[9]

The analemmatic sundial we know today was probably invented some time in the period[10] between 1532 and 1640. The timing could not have been more unlikely for the introduction of a modern sundial based on an ancient analemma:

> [The ancient] type of dial has fallen into disuse, since we stopped dividing the day into temporary hours. The Ptolemaic theory would therefore be perfectly useless to us today, if his constructions could not be equally adapted to the new system.... When the book of the Analemma was published for the first time by Commandin, in 1562, gnomonics had already been founded on totally different principles. See the *Horologiographia* of Munster, of which the first edition is of 1531, and the second of 1533. [11]

Whoever invented the dial managed to combine the three senses of *analemma*[12] into a single accomplishment which not only bridged the centuries but transformed an old concept so that it made sense in a world of equal hours - the new paradigm of time measurement.

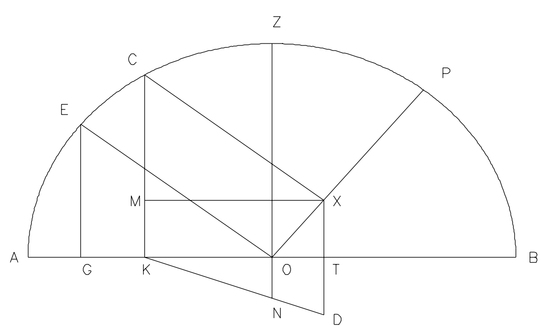
> [The analemma was] applicable to the ancient dials which , as everyone knows, have their style perpendicular to their face. It had lost all practical utility with the modern dials, based since the 15th century on the inclination of the style parallel to the axis of the world. But the analemmatic dial, with perpendicular style, appearing in the texts of the 17th century, revived the use of the analemma through its geometric construction. [13]

The analemmatic dial is little else than the *graphical procedure* we know as *orthographic projection* turned into an *instrument* to tell time. Its ellipse of hour-points results from an orthographic projection of the sun's path from the pole onto the horizon circle. Authors ranging from Vaulezard[14] in 1640 to Lalande, more than a century later, derived its distinctive declination scale for the placement of the vertical gnomon directly from the traditional analemma drawing.

The 18th century astronomer J.J. de Lalande was born in Bourg-en-Bresse (Ain), France, near the church of Brou, on whose grounds is found the oldest analemmatic sundial in existence.[15] In 1756 he was concerned that the dial was deteriorating, so he had it reconstructed in stone at his own expense. The next year, he communicated to the Académie des Sciences a letter which identified the task of drawing and justifying an analemmatic dial as "one of the most complicated in all of Gnomonics."[16] Lalande complained that earlier treatments of the dial either gave no demonstration that it was correct or gave an unsatisfactory, incomplete demonstration. Although his survey of the prior literature was far from complete,[17] Lalande noted that he was forced to find his own demonstration of the correctness of the dial. In doing so, he "formed an extremely simple rule for finding the place of the style in all the months of the year."[18] But what was the basis of this rule? As he noted himself, it was the ancient analemma.

J.J. de Lalande

This dial has been called analemmatic because it is traced by means of the figure known as the *analemma*: [19]



This figure[20] shows an orthographic projection onto the meridian plane AZB. The horizon is AB, the pole P, the zenith Z, EO the projection of the equator, and CX the projection of the minor circle representing the sun's orbit on the given day. By selecting point D so that the lengths KD and GO are equal, Lalande shows that ND is the correct displacement (*i.e.* $\cos\varphi \tan\delta$) of the vertical gnomon or style from the dial center for the given day.

The analemmatic sundial thus relies on the ancient analemma to divide the day into modern solar hours. But what does this have to do with the analemma curve? So far, nothing. Our next task, then, is to consider the modern meaning of *analemma*.
We begin by returning once again to the ancient analemma, which generally was drawn as a projection onto the meridian plane, thus lending itself to the representation of noon time shadows.

> The analemma's relation to dialling is evident in the first step of its construction. Two perpendicular lines are drawn to represent a gnomon... and its equinoctial noon shadow. [21]

By projecting the daily circles of the sun's path onto the meridian plane, the analemma naturally registered the various positions of the noon time sun by the length of the gnomon's shadow. Implementing this graphic scenario as an instrument or analemma would be a natural evolution. Indeed, we find reference to such an ancient instrument at least as early as 1753:

> *Analemma* in ancient writers denotes those sort of sun-dials which shew only the height of the sun at noon, every day, by the largeness of the shadow of the gnomon. [22]

If, then, *analemma* once referred to an instrument which focused on the recording of noon time shadows, the transition to an instrument which does the same under the "new" definition of the time is an easy one to make.

The analemma curve which is familiar to us today was conceived in 1740 by Jean Paul Grandjean de Fouchy, secretary of the *Académie des Sciences* in Paris.[23] The curve provided a graphic representation of the equation of time and was intended for use on meridian lines; indeed, it was called the *meridienne de temps moyen* (mean time meridian).[24] As such, it was clearly useful in the design of sundials, providing an easy graphical method of showing the difference between apparent and mean time. The curve was not restricted to the meridian, and appropriate variants of it found their way to each hour-line of the classical sundial. Consider, for example, the 1876 sundial by Father Ildéphonse at the convent of Cimiez-sur-Nice in France.[25] This dial superimposed an analemmatic curve on each hour-line; the dial carried the legend *Temps Vrai et Temps Moyen* (True Time and Mean Time). Now, more than a century later, this technique still plays a prominent role in many of the modern sundials available for purchase today.[26] As early as 1826 the analemma curve was incorporated into a dial by the Abbot Guyoux as the target for a spot of light;[27] this approach was improved and patented by Paul Fléchet[28] in 1860 and 1862. Just five years later, on 21 May 1867, Lloyd Mifflin was granted the first U.S. Patent[29] for a gnomon incorporating the shape of the analemma curve. The innovation of shaping the gnomon so that it reproduces the analemma and registers mean time on an equatorial hour-ring has since appeared in many popular sundials; in 1966 the three winners of a *"Sundial of the Year 2000"* contest all benefitted from this device.[30]

And yet, analemma curves are rarely to be found on an analemmatic sundial. Our etymological investigation has tracked them both to a common ancestry; we need now to effect a reunion.

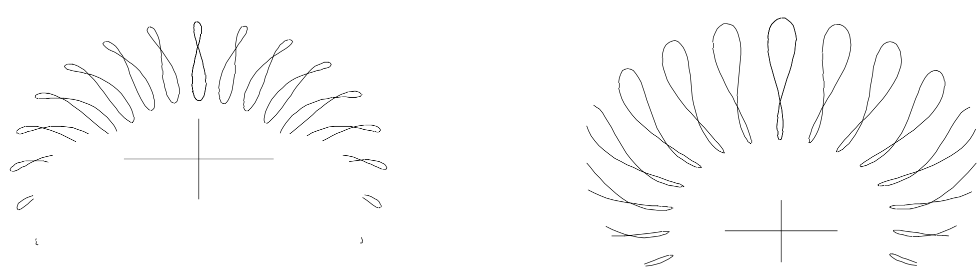
Polyanalemmatics

In 1970 Hermann Egger of Zurich Switzerland published a brief description[31] of one method for adapting the analemmatic sundial to register mean time. The resulting dial, which Egger referred to as a polyanalemmatic, associates each of the hour-points on the elliptical hour ring with an analemma curve. Egger eliminated the declination scale along the meridian line and arranged the analemmas so that if, at mean time *T*, the vertical gnomon is placed at the appropriate date on the analemma curve for *T*, then the gnomon's shadow will fall on the *T* hour-point.

As Egger suggested, this dial might be of primary interest as a large installation in a public park, where passersby might be enticed to walk among the analemmas to find the correct one to use by observing how close their shadows come to the hour-point associated with the curve. Unfortunately, this arrangement does not lend itself well to an instrument to be used as a sundial. Instead of being able to position the gnomon and then watch the time pass by as its shadow progresses through the hour-points, this dial effectively requires a new placement of the gnomon every hour; and the dialist may try a number of the analemmas before discovering the right one to choose. It is also very difficult to obtain any reasonable reading between the hour-markers, since the gnomon is properly in two different positions when it records the two times at either end of the interval and should in fact be at neither of those positions in the interim.

[1]

Polyanalemmatics for London



$F_S = 1.25 \quad F_W = .625$ $\qquad$ $F_S = 1.0 \quad F_W = 2.0$

A more useful dial results if the model provided by Father Ildéphonse, as noted above, is followed more closely. To accomplish this, replace each hour-point by an analemma curve, and retain the normal declination scale on the meridian line. In this way, the vertical gnomon is stationary throughout any given day and time is read by the intersection of its shadow with the proper date points on the analemma curves representing the hours. In operation, this is similar to the Ildéphonse dial. The additional requirement of moving the gnomon each day is a trade-off for the resulting convenience of not having to distinguish the exact end-point of the gnomon in order to read the time - the intersection of the shadow and the curve suffices.

Given this general description, there are many different ways in which the analemmas can be calculated and arranged. Two examples are given here[32] in Figure **[1]**.

In the equations **[2]** used to produce these configurations, the values of F_S and F_W, chosen to be constant for each dial, significantly affect the layout. A value of 1.0 for F_S will place the summer solstice date for each analemma curve at approximately[33] the same location the hour-point would have occupied on the traditional elliptical hour ring. A greater value moves this date farther out on the defining shadow, and a smaller value brings it within the perimeter of the ellipse. The choice of a value for F_W has a similar

[2]

Given constants F_S and F_W for a polyanalemmatic sundial and the equation of time ε_δ depending on the solar declination δ, determine the coordinates (x_δ, y_δ) of the analemma curve for mean time T.[34]

***

$$S = .5 \times \sqrt{(\cos T \sin\varphi - .44343 \cos\varphi)^2 + \sin^2 T}$$

$$W = .5 \times \sqrt{(\cos T \sin\varphi + .44343 \cos\varphi)^2 + \sin^2 T}$$

$$D_\delta = SF_S + WF_W + 2.30718 \tan\delta (SF_S - WF_W)$$

$$\cot Z_\delta = (\cos(T - \varepsilon_\delta)\sin\varphi - \tan\delta\cos\varphi) / \sin(T - \varepsilon_\delta)$$

$$x_\delta = D_\delta \sin Z_\delta \qquad y_\delta = D_\delta \cos Z_\delta + \cos\varphi \tan\delta$$

impact on the placement of the winter solstice date for each analemma. The equations then position all intervening dates at appropriate points so that at apparent time $T - \varepsilon_\delta$ the shadow of the gnomon will correctly intersect the analemma corresponding to mean time T.

Unfortunately, this dial continues to suffer from the difficulty encountered when trying to read the time between analemmas. It is possible to add more curves, but they quickly begin overlapping and the decorative dial becomes more and more confusing. We will look elsewhere for a more practical approach.

Experiments With Analemmas

The ancestor of all existing analemmatic sundials, the dial which Lalande rebuilt in 1756 at the church of Brou, was restored once again in 1902. The craftsman charged with its restoration was an amateur dialist who decided to overlay an analemma curve on the meridian line. The dates were not marked on a linear declination scale; they were noted only on the curve itself. This situation quickly resulted in the common belief that the vertical gnomon or erect visitor should be stationed on the curve rather than the meridian in order to cast a shadow on the hour-ring.[35] There is of course no justification for this arrangement; it introduces significant error into the dial design.

In Kennett Square, Pennsylvania USA, not far from Philadelphia, there is a 1050 acre horticultural park, Longwood Gardens.[36] The park is on the former country estate of Pierre S. duPont (1870-1954). The park's last construction project overseen by duPont was the design of a 37 by 24 foot analemmatic sundial in what is now a Topiary garden in the park. The dial was completed in 1939 after more than six years of daily noon-time observations:

> After about eight months of trying to use [calculated] measurements, Mr. duPont got disgusted and said we would build the sundial ourselves after working out our own measurements. Both Roland Taylor and I [Knowles R. Bowen] began the task of taking actual observations on the sun at 12 noon every day, after checking our time with the Naval Observatory at Annapolis. If the sun was not out, we could not get that day's sighting until the following year, or maybe two years later, which explains why the project took so long. [37]

At some point during this long process, a visit was made "to France to check out a sundial there; we copied some details, but the dial was not too accurate...."[38] - presumably this was the dial at Brou. Some years later, in 1946, duPont commissioned additional work on the dial, to reposition the hour-markings on the ellipse. One can only wonder if the purpose behind this change was to attempt to correct the errors which no doubt were becoming obvious in the dial readings. The errors would have resulted from placing a standard analemma at the center of an analemmatic sundial, thus perpetuating the design flaw begun at Brou - a flaw which could not be corrected by a change in the hour-markings.

In the late 1960's another analysis was done on the Longwood Gardens dial, enlisting the assistance of P. Kenneth Seidelman of the U.S. Naval Observatory. Measurements showed that the hour-points were positioned for standard time readings, but the double analemmas then at the center of the dial[39] proved to be little more applicable to a proper design than a single analemma. To address this problem, Seidelman developed a weighted average approach to defining substitute analemma curves which results in a close approximation to mean time.[40] The engraving of a new pair of analemmas was undertaken in 1978, and the Longwood Gardens dial was thus corrected in a novel way. The remainder of this article is dedicated to developing the mathematics behind the Longwood Gardens adaptation.

<u>Standard Time Analemmatic Sundial</u>

We begin by noting that the real challenge here is to move from apparent time to mean time. Since, for any location, the difference between standard and mean times is constant; the additional shift to standard time is simply a matter of relabeling the hour-points.

Equation **[3]** specifies the line of shadow which at mean time *T* will intersect the hour-point for *T*. Note that for the special case in which we ignore the equation of time (*i.e.* set $\varepsilon_\delta = 0$), dealing then only with apparent time, this equation collapses to the usual analemmatic sundial with all of the shadow lines for a given day intersecting the meridian

[3]

Determine the equation of the shadow of a vertical gnomon at mean time T, if the base of the gnomon is placed on the elliptical hour-ring at the point corresponding to time *T* (*i.e.* at point $x_T = \sin T$, $y_T = \cos T \sin\varphi$).

***

$$y\sin(T-\varepsilon_\delta) = x(\sin\varphi\cos(T-\varepsilon_\delta) - \cos\varphi\tan\delta) - \sin\varphi\sin\varepsilon_\delta + \cos\varphi\tan\delta\sin T$$

at the same point, $(0, \cos\varphi\tan\delta)$. However, once the equation of time enters the picture, the shadow lines for any given day no longer all converge. Instead, they all become tangent to a single curve (equations **[4]**), technically known as their envelope.

[4]

Determine the envelope of the family of shadow lines which intersect the elliptic hour-ring at appropriate mean times *T*. [41]

***

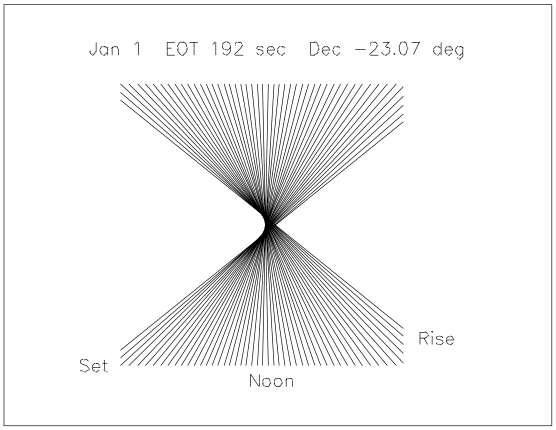
$$t = T - \varepsilon_\delta$$

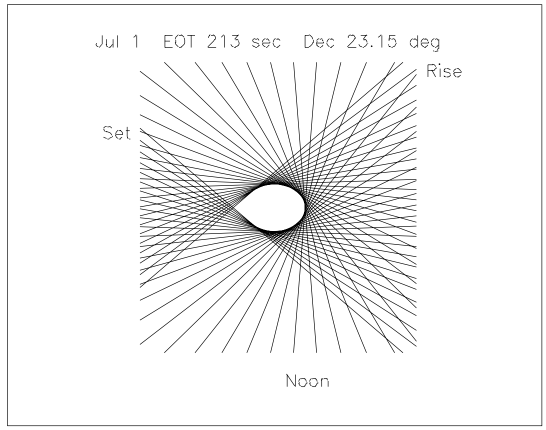
$$x = \sin\varepsilon_\delta \frac{\sin\varphi\cos t - \cos\varphi\tan\delta}{\sin\varphi - \cos\varphi\tan\delta\cos t}$$

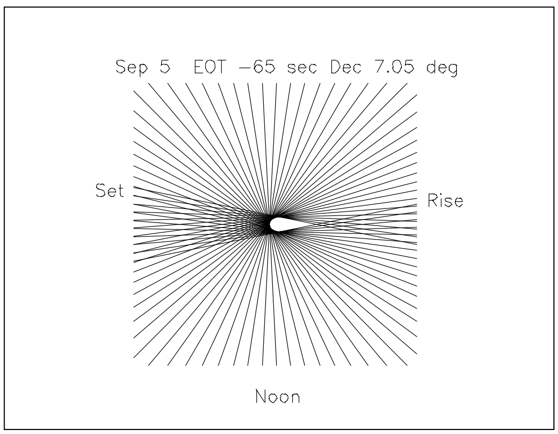
$$y = \frac{\cos\varphi\tan\delta[\sin\varphi\cos\varepsilon_\delta - \cos\varphi\tan\delta\cos T] - \sin^2\varphi\sin\varepsilon_\delta\sin t}{\sin\varphi - \cos\varphi\tan\delta\cos t}$$

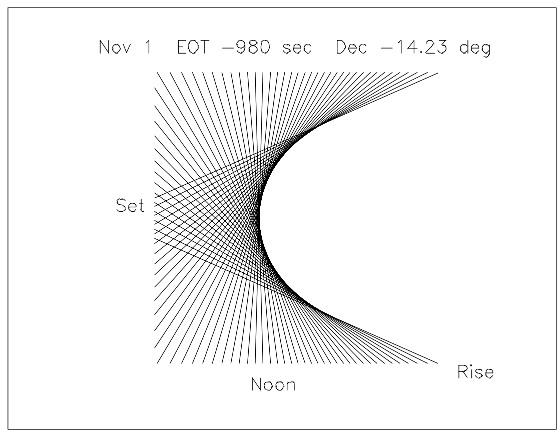
[5]

Envelopes of Shadow Lines Registering Mean Time on the Elliptic Hour Ring

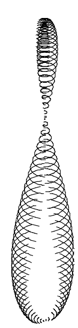








Analemma of Envelopes [6]



Overlay of Analemmas [7]

The shape and orientation of the envelope vary from day to day; examples are given in figure **[5]**, where selected shadow-lines ranging from sunrise through noon to sunset are shown, all tangent to their envelope. On days for which the solar declination is positive (assuming a location north of the equator), this parade of lines swings through at least 180° and in effect seals or closes the envelope; on days with a negative solar declination, the envelope remains open. Note also that a large absolute value for the equation of time results in a wider envelope. If the equation is positive, the envelope opens to the west; if negative, the opening is to the east.

If we now plot the envelopes for each day throughout the year (figure **[6]**) in their correct positions relative to each other, we find, not surprisingly, that they come together to form a figure 8 analemma.[42] However, even though we make the usual assumption that the solar declination is constant on any given day, we do not have discrete points representing the correct gnomon position for each day; instead, we have small curves. To solve this dilemma, we will select average points to replace the curves, with the selection being made to minimize the error introduced by the averaging. Since experimentation shows that use of a single point for each day would result in no advantage over a traditional dial with no analemma at all,[43] we will replace each day's envelope by two distinct points, one selected for use in the morning hours and one for the afternoon.

There are a number of approaches we can take at this juncture; we will briefly consider four options: **I-IV**, and the decision on which option to use is ultimately up to the individual dialist.

I.

Select a number of evenly-spaced *apparent* times (t) between noon and the time of sunset on the winter solstice. Find weighted averages of the coordinates of the points (x_t, y_t) on the envelope for the given date and associated with the selected times (equations **[8]**). To understand the rationale for the weightings to be used, consider two shadows: one falling in a North-South direction, and the other East-West. The placement of the gnomon casting the North-South shadow is very sensitive to the x coordinate of the point at its base, but is totally independent of the y coordinate. A change in the y coordinate will not change the line on which the shadow lies, but any change in the x coordinate will place the shadow on a totally different line. Similarly, for an East-West shadow, the gnomon's placement is sensitive to the y coordinate and insensitive to the x coordinate. For shadows between these two, sensitivity to either coordinate will depend on the slope of the shadows; use of the sines and cosines of the shadow's azimuths as weights reflects this sensitivity.

This process is then repeated for the morning hours. The result is a pair of similar but not identical analemmas, as can be seen in figure **[7]**, where such a pair, calculated for London, is shown with one curve overlaying the other.[44]

[8]

For a given date with solar declination δ , and selected apparent times *t*, with corresponding points (x_t, y_t) on the envelope of the shadow-lines, determine the weighted average coordinates of the date's analemma point (morning or afternoon).

***

$$\cot Z_t = (\cos t \sin\varphi - \tan\delta \cos\varphi) / \sin t$$

$$x = \left(\sum_t x_t \left|\cos Z_t\right|\right) \Big/ \sum_t \left|\cos Z_t\right| \qquad\qquad y = \left(\sum_t y_t \sin Z_t\right) \Big/ \sum_t \sin Z_t$$

To implement the dial, we need only draw the two analemmas side by side and split the elliptical hour-ring at noon, with the space between the two halves equal to the space between the North-South axes running through the analemmas. The vertical gnomon is placed each morning on the appropriate date point of the west analemma, and at noon it is moved to the corresponding point on the east analemma. To adjust the dial from mean to standard time, simply relabel points on the ellipse to reflect the constant difference between standard and mean times.

II.

To consider a second option, observe that if the intersection of the shadow-lines of any two of the selected apparent times (*t*) were used as the analemma point (equations **[9]**), then there would be no error in the dial reading at those two times. Indeed, if we selected an analemma point *within* the envelope, we would have no correct readings during the day; a point selected *on* the envelope produces only a single correct reading. But a point

[9]

Determine the point of intersection $({}_1x_2,\ {}_1y_2)$ of the shadow-lines for mean times T_1 and T_2 corresponding to apparent times $t_1 = T_1 - \varepsilon_\delta$ and $t_2 = T_2 - \varepsilon_\delta$.

***

$$M_\delta = \cos\varphi \tan\delta$$

$${}_1x_2 = {}_2x_1 = \sin\varepsilon_\delta \frac{\sin\varphi(\sin t_1 - \sin t_2) - M_\delta \sin(t_1 - t_2)}{\sin\varphi \sin(t_1 - t_2) - M_\delta(\sin t_1 - \sin t_2)}$$

$${}_1y_2 = {}_2y_1$$

$$= \frac{M_\delta[\cos\varepsilon_\delta \sin\varphi \sin(t_1 - t_2) - M_\delta(\sin T_1 - \sin T_2)] + \sin\varepsilon_\delta \sin^2\varphi(\cos t_1 - \cos t_2)}{\sin\varphi \sin(t_1 - t_2) - M_\delta(\sin t_1 - \sin t_2)}$$

selected from a small region just outside the envelope, the region of intersecting shadow-lines which shows in figure **[5]** as cross-hatching, gives completely correct readings twice per day. Perhaps these are the points which should enter into the weighted average. Continuing the restriction that the apparent times (*t*) are either all morning or all afternoon times, the region of interest is further limited to the area bounded by the envelope, the noon shadow-line, and either the sunrise or sunset shadow-line.

If we select *N* apparent times, there are $N^2 - N$ points of intersection of pairs of shadow-lines. To determine the points of the analemma, we take a weighted average of the coordinates of these intersections (equations **[10]**). The dial is then completed as in the earlier option, by splitting and perhaps relabeling the hour-ring and setting the gnomon appropriately for morning and afternoon.

[10]

For a given date with solar declination δ , N selected apparent times *t*, and the $N^2 - N$ pairwise points $\left({}_j x_k \, , \, {}_j y_k \right)$ of intersection of the corresponding shadow-lines, determine the weighted average coordinates of the date's analemma point (morning or afternoon).

***

$$\cot Z_k = \left(\cos t_k \, \sin\varphi - \tan\delta\cos\varphi\right) / \sin t_k$$

$$x = \left(\sum_j \sum_{k \neq j} {}_j x_k \left|\cos Z_k\right| \right) \Big/ (N-1) \sum_k \left|\cos Z_k\right| \qquad y = \left(\sum_j \sum_{k \neq j} {}_j y_k \sin Z_k \right) \Big/ (N-1) \sum_k \sin Z_k$$

III.-IV.

As a third and fourth option for the definition of the two analemma curves, consider broadening the interval from which the apparent times (*t*) are selected. By using the interval between noon and sunrise/sunset on the specific given date, we decrease both the overall average and the absolute maximum deviation from mean time. The tradeoff is that the average deviation in the hours when the sundial is probably most often used goes up and the maximum error at noon increases.

Obviously, an important element in the evaluation of these various options is a measure of the error or deviation from mean time that they incur. Equations **[11]** provide a means for determining the apparent time at which the gnomon's shadow crosses any given hour point. Following this calculation, a natural measure of the error is to consider the difference between the time (*T*) registered by the shadow on the dial, and the actual mean time equal to the sum of the apparent time (*t*) at which the reading is made and the day's equation $\left(\varepsilon_\delta\right)$.

[11]

Given a vertical gnomon with base at point (x_δ, y_δ), determine the local apparent time(s) t at latitude φ and solar declination δ at which the shadow of the gnomon will pass through the point (x_T, y_T).

***

$$a = y_T - y_\delta \qquad b = (x_\delta - x_T)\sin\varphi \qquad c = (x_\delta - x_T)\cos\varphi\tan\delta$$

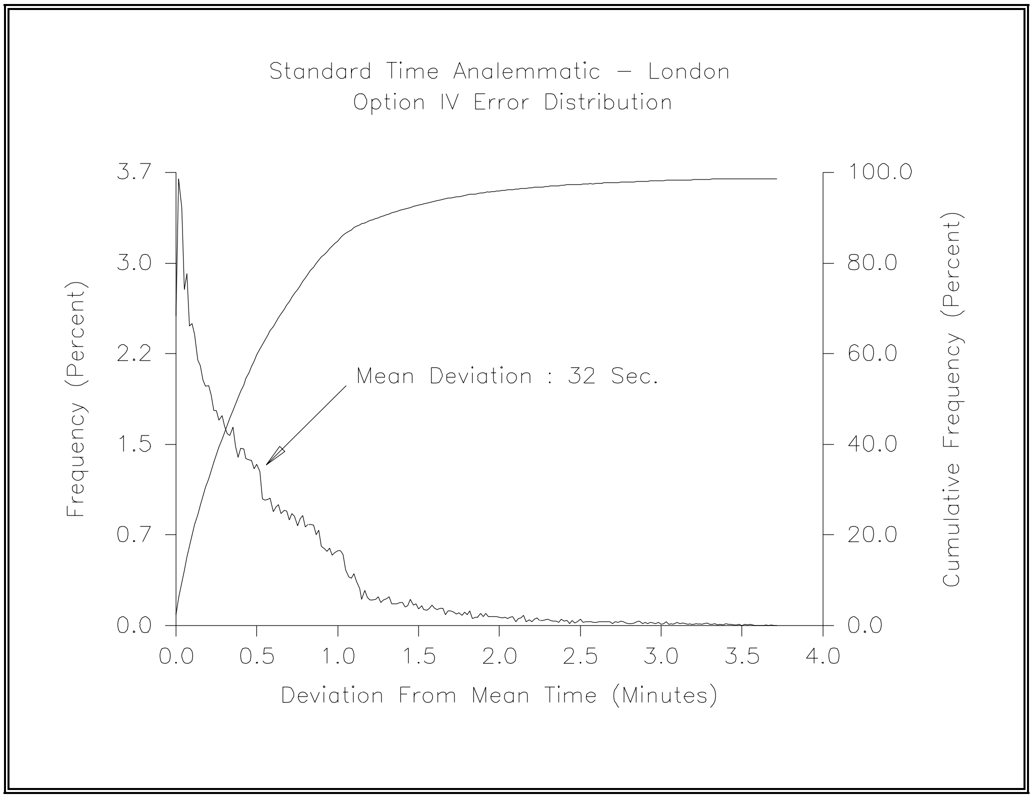
$$\sin t = \frac{ac \mp b\sqrt{a^2+b^2-c^2}}{a^2+b^2} \qquad \cos t = \frac{bc \pm a\sqrt{a^2+b^2-c^2}}{a^2+b^2}$$

The deviations (in seconds) from mean time incurred by dials designed for London according to these four options appear in the following table: [45]

	I	II	III	IV
Average Deviation	44	46	38	32
Maximum Deviation	399	410	370	223
Average Deviation (Noon ± 3.7 Hours)	15	14	31	30
Average Noon Deviation	49	40	87	80
Maximum Noon Deviation	120	99	203	198

Thus, if we choose option IV, by introducing a pair of analemmas we produce an analemmatic sundial which always registers within 4 minutes of mean time and on average errs by 32 seconds, little more than one-half minute. Although the deviation at mean noon, the time of shifting from one analemma to the other, can exceed 3 minutes, the average error when the dial records this time is within the limits of our ability to distinguish different readings on the dial. These values should be compared with the corresponding numbers for an uncorrected dial, with no analemmas; such a traditional dial located in London would have an average deviation from mean time of 385 seconds[46] and, of course, could suffer from an error in excess of 16 minutes.

A better sense of the deviations for option IV can be obtained from the following more graphic presentation, which also shows that for 90% of the time that the sun is above the horizon the dial's deviation from mean time is less than 69 seconds:



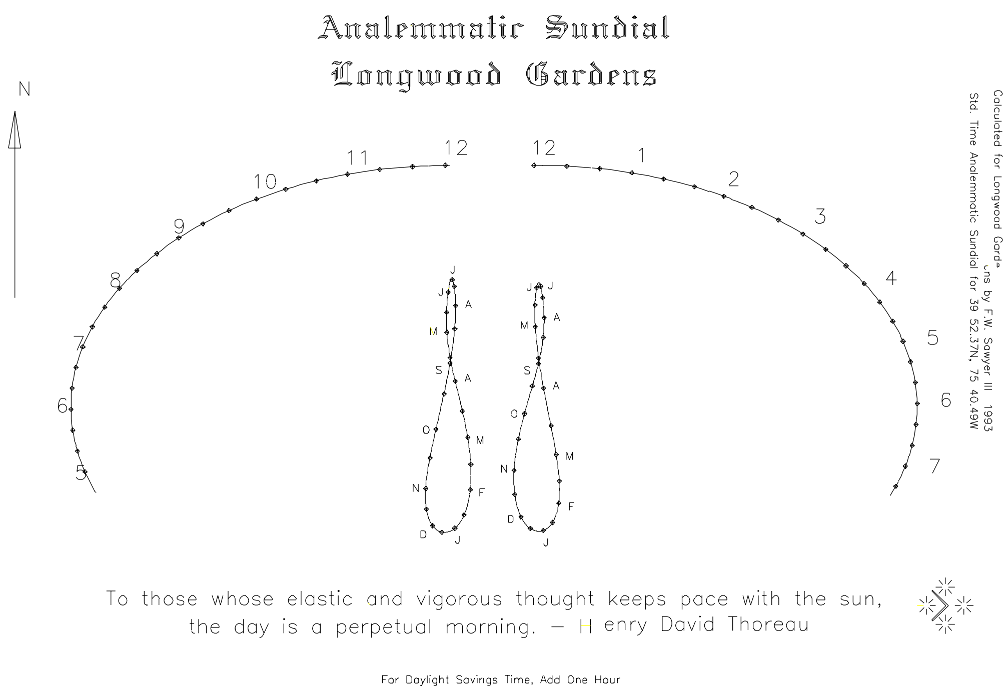
Reunion & Sources

Having presented the requisite formulas, we now see that the stray modern sense of *analemma* - with perhaps a slight evolution in its meaning - can be reunited with the analemmatic sundial that already incorporates all of the prior historical senses of the word. The result is a dial which can provide a fair reading of standard time, yielding a level of accuracy well within the tolerance of most dialists.

To assure that a reading can be taken at all times that the sun is above the horizon, the usual minimum value for the height of the gnomon is $1.09\cos(\varphi - 23.43°)$ times the semi-major axis. This formula continues to be useful for the standard time analemmatic, only slightly understating the required height.[47]

Similarly, in order to assure that the gnomon's shadow is sharp as it intersects the ellipse, the width of the gnomon should be approximately one percent of the semi-major axis of the dial.[48]

For the dialist interested in experimenting with this combination, a sample dial calculated for Longwood Gardens (more specifically, for 39.87ºN, 75.67ºW) is presented here.



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[1] This article assumes a basic familiarity with the traditional analemmatic sundial. On a horizontal plane with coordinates pointing to the cardinal points, x increasing to the east, and y increasing to the north, the dial point corresponding to apparent time t is $(\sin t, \cos t \sin\varphi)$. These points lie on an ellipse with an east-west semi-major axis of 1 and a north-south semi-major axis equal to $\sin\varphi$, where φ is the latitude of the dial. The gnomon is a vertical rod which is positioned daily at the point $(0, \cos\varphi\tan\delta)$, where δ is the solar declination for the given day.

[2] Delambre 1817, p. 458. English translations of excerpts from Delambre, Janin and Lalande, here and elsewhere in the present paper are by the author. Note the italicized *taken up*; the etymology of the word is ultimately from the Greek *ana* (up) and *lemma* (taken). Louis Janin alludes to the same sense of *analemma* as does Delambre: "en employant alors ce mot, comme beaucoup d'anciens auteurs, dans le sens de procédé de résolution graphique." Janin 1974, p. 8.

[3] The reference is to *De Architectura 9.8*: Ex quorum libris, si qui velit, subiectiones invenire poterit, dummodo sciat analemmatos descriptiones.

[4] Gibbs 1976, pp. 107-108 cites the work of modern commentators such as Gustav Bilfinger and Joseph Drecker who have provided detailed instructions for the use of the orthographic projection in the design of specific types of sundials. For a more recent study culminating in the development of a horizontal sundial showing temporary hours, see Drinkwater 1993.

[5] For a discussion of Ptolemy's analemma see Gibbs 1976, pp. 109-117, and Delambre 1817, pp. 458-503. Unfortunately, not all of Ptolemy's work survives, so even this more detailed discussion of the analemma does not yield specific instructions for dialling. Thus: "nowhere in the preserved part of the analemma does Ptolemy explain why his six angles...might be of interest to diallers." Gibbs 1976, p. 116.

[6] A carpenter's square is the etymological origin of the word *gnomon*.

[7] See Delambre 1819, p. 326.

[8] See Delambre 1819, p. 330.

[9] See Delambre 1819, p. 626. The reference is to Valentin Pini's publication of *Fabrica de gl' horologi solari* in Venice.

[10] The oldest texts on the analemmatic sundial are Vaulezard 1640 and 1644; the earliest treatment in English is Foster 1654. Foster also considers a number of interesting variants, including a dial with nonvertical gnomon and equispaced hour-points, and a retrograde dial whose hour-points are all on a finite segment of a single straight line; see Sawyer 1991 and 1992. The oldest known analemmatic is the famous dial at the church of Brou, in Bourg-en-Bresse (Ain), France. Virtually every discussion of this topic in the literature makes an obligatory reference to the Brou dial. Verbal tradition suggests that the dial is as old as the church, which dates from 1506; but there seems to be no real evidence for the tradition. Janin 1974, p. 14 points out that the first written mention of the dial does not occur until the 18th century. It would seem more likely that the dial, which was originally in the cemetery of the church, was not constructed until after the 1532 publication in Paris of Oronce Finé's *Protomathesis* which included a comprehensive treatment of dial types but no mention of the analemmatic sundial. Finé's treatment of dialing was published separately in 1560 as *De Solaribus Horologiis*, and has recently been paraphrased in English by Drinkwater 1990.

[11] Delambre 1817, p. 472.The first reference is to Frederico Commandin's publication of Ptolemy's work in Rome. The second is to Sebastian Munster's *Compositio Horologiorum* 1531 and its renamed second edition *Horologiographia* 1533; this book was the first publication surveying all types of dials showing equal hours and having a gnomon parallel to the celestial axis.

[12] The Oxford English Dictionary (first edition) gives the following etymology: L. *analemma*, the pedestal of a sun dial, hence the sun-dial itself Gr. αναλημμα, a prop or support , from αναλαμβ–αν–ειν, to take up, resume, repair. Thus the word itself, as borrowed from the Greek, is intended to suggest the idea of a support. The reported etymology takes this suggestion very literally (*i.e.* a pedestal); while the sense to which we have alluded is more figurative, with the analemma providing graphical support to shorten an otherwise more difficult construction. Going beyond the etymology, the OED gives four definitions of *analemma.* One of them corresponds to the figure 8 graph to which we will turn presently; a second is simply "a sort of sundial". The remaining two correspond to the other senses we have explored here:"orthographic projection...as used in dialling"; and "a gnomon or astrolabe [*i.e*. an instrument]...used in solving certain astronomical problems."

[13] Janin 1974, p. 3.

[14] Vaulezard 1644. For a historical review of the different proofs for the analemmatic dial, see Janin 1974.

[15] See Note 10 above.

[16] Lalande 1757, p. 483.

[17] He did not find either Vaulezard or Foster.

[18] Lalande 1757, p. 483.

[19] Lalande 1757, pp. 484, 488.

[20] The figure is virtually identical to the one given in Vaulezard 1644, which is reproduced in Janin 1974, p. 7. The construction was cited by many authors, even after simpler constructions were found.

[21] Gibbs 1976, p. 105.

[22] See the Oxford English Dictionary (1st ed.) entry for *Analemma*, which cites this passage to support its listing of "a sort of sundial" as the first meaning of the word.

[23] Attributions appear in Delambre 1819, p.637; Apel 1990, p. 66; and Boursier 1936, p. 175. A more detailed discussion of the context of and the evidence for this invention are given in Gotteland 1990.

[24] What is still missing at this point is the first historical connection of the word *analemma* with Grandjean de Fouchy's mean time meridian. The author would be pleased to have such a citation. In French, the curve is still the meridienne de temps moyen; the German and Italian words are *Zeitgleichungskurve* and *Lemniscata*, respectively. English seems to be the only language linking the curve to etymological roots in the ancient analemma.

[25] For an illustration of this dial, see Boursier 1936, p.174. Pictures of this dial, based on the Boursier illustration, have appeared in a number of dialling books.

[26] Examples of firms specializing in producing sundials of this variety are Ameco Sundials of Poulsbo, Washington USA and Celestial Arts & Sciences of Santa Fe, New Mexico USA. The latter firm refers to its offerings as analemmatic sundials, when they are in fact classical dials with analemma curves replacing the hour-lines in a manner similar to the Ildéphonse dial mentioned in the text of this article. The usage of the *analemmatic sundial* terminology in virtually every modern text on dialing would suggest that its application to a classical dial, based on a gnomonic rather orthographic projection, is either incorrect or it heralds a significant shift in dialing terminology.

[27] Cited in Apel 1990, p. 67.

[28] U.S. Patent #35,225 issued 13 May 1862 to Paul Fléchet; Apel 1990, p. 67.

[29] U.S. Patent #64,982 issued 21 May 1867 to Lloyd Mifflin.

[30] The contest was conducted by Hermann Egger in the pages of *Sky & Telescope* magazine. See Egger 1966.

[31] Egger 1970.

[32] The omitted portions on some of the analemma curves in these illustrations correspond to dates and hours at which the sun is below the horizon.

[33] The placement is approximate because we have simplified the formulas for S and W by using T instead of $T-\varepsilon_\delta$.

[34] Note that, as given, these formulas do not uniquely determine the value of *Z*. However, since *Z* represents the solar azimuth, its value must range between -180° and +180°, and it must have the same sign as the apparent time $T-\varepsilon_\delta$. With these additional conditions stipulated, *Z* is uniquely determined.

[35] For a discussion of this situation and the indignation it created among dialists, see Janin 1970.

[36] The author is indebted to Colvin L. Randall of the Longwood Gardens staff for his help in providing information on the park and on the history of the Longwood Sundial.

[37] Thompson 1976, p.93, quoting recollections of engineer Knowles R. Bowen.

[38] Thompson 1976, p.94.

[39] It is not clear when the double analemmas became part of the sundial. The published recollections of Knowles Bowen, the engineer, and comments by George Thompson, who was duPont's personal secretary (Thompson 1976), seem to suggest a single analemma on the original dial; this view is supported by the reports of noon-time readings being taken. However, no one seems to be able to recall a time when the dial did not have a double analemma arrangement, and there is a photo in the Thompson book (p. 94) of Bowen with two gnomons astride two analemmas; the photo is undated.

[40] Seidelman 1970 and 1975. Developing on this idea, a large sundial fountain with weighted-average double analemmas was designed by Albert M. Thorne for a mall area at the University of North Carolina at Charlotte. Unfortunately, funding was not obtained for the sundial fountain, and the Longwood Gardens dial remains as perhaps the only large standard time analemmatic sundial.

[41] The easiest way to derive this equation is to apply l'Hôpital's rule to determine the limiting values for equations **[9]** as $t_2 \Rightarrow t_1$.

[42] Seidelman's only comment on this collection of envelopes is "The hourly positions of the gnomon are a helical curve, whose daily average is shaped like the figure eight" (Seidelman 1970, p.4). If we treat the solar declination δ as a continuous quantity and calculate the envelope curve as well for times when the sun is below the horizon, we do in fact obtain a helical curve which winds itself into a figure 8. Note that a case can be made for the view that applying the term *analemma* to this and later curves discussed in this article amounts to an extension or further generalization of its meaning. For purposes of clarity, the illustration given here actually only shows the envelopes for every third day throughout the year.

[43] For a London dial, use of a single analemma - one point for each day - would result in an average error in excess of 3 minutes, with extremes ranging as high as 16 minutes, and a variation within one day of as much as 12 minutes. The usual spatial symmetry of the analemmatic sundial about the meridian fails when we introduce the equation of time. Accounting for the equation requires nudging shadow lines a little closer to the meridian line on one side and a little farther away on the other. Unfortunately, the equation of time does not change sign when the sun crosses the meridian.

[44] Although none of the options presented here matches exactly the approach adopted by Seidelman 1970; this first option is perhaps the closest. Seidelman limited his average to the time interval corresponding to sunlight on the winter solstice day (1970, p.5), but the points selected to go into the average were not exactly on the envelope. He used the intersections of the shadows for successive hour-points, and he apparently dealt only with full hour intervals (p.4). If more points were selected, the interval between successive times would decrease, and the resulting points would all approach the envelope as a limit. Thus, this option may be viewed as Seidelman's 1970 technique applied to a larger number of points. Note that in Seidelman 1975 there is an indication that a wider interval was actually used for the Longwood Gardens dial, thus suggesting that the final implementation may have corresponded more to Option II.

[45] A similar table for the Longwood Gardens site (39.8728ºN, 75.6748ºW) is given here:

	I	II	III	IV
Average Deviation	40	39	38	32
Maximum Deviation	359	341	266	216
Average Deviation (Noon ± 4.5 hours)	21	20	28	28
Average Noon Deviation	53	48	80	77
Maximum Noon Deviation	132	112	205	201

Seidelman 1970 states "The average error for any day is less than one minute. The maximum error for times included in the average is 2.71 minutes for 7 am and 2.12 minutes for noon." (p.6). The times Seidelman uses are limited to 5 hours before and after noon. He gives a table which also shows the error at 6 am growing to 4.35 minutes, or 261 seconds. The maximum deviation shown in the table above occurs outside the interval considered by Seidelman.

[46] This value is an average of the absolute values of the equation of time, weighted by the duration of sunlight in London on the respective dates.

[47] For London, the formula produces a height of 0.9615 times the semi-major axis. Actual calculation for the analemmas illustrated here shows that, for the standard time dial, at 12:05pm on Jul 3 the gnomon height should be 0.979 times the semi-major axis. A weakness in most large analemmatic dials is that they seldom can incorporate gnomons tall enough to satisfy this formula; the Longwood Gardens gnomon would have to be more than 19 feet tall.

[48] Janin 1974, p.26.